Minnesota STEM
Teacher Center
Frameworks for Science & Mathematics Standards

Use and Connect Mathematical Representations

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In the previous fall issues of *MathBits*, various aspects of the new professional development materials found on the Minnesota STEM Teacher Center website have been explored. As a reminder, these new materials are organized in four clusters: Cluster One: Planning & Assessment, Cluster Two: Instruction, Cluster 3: Language development, and Cluster Four: Differentiation. This month we return to Cluster Two and explore the instructional practice of using and connecting mathematical representations produced during a lesson. This instructional practice serves as a focal point for both how local and whole class sharing of student thinking unfolds. In the posing of questions to elicit student thinking, the various representations produced by students in the classroom become key elements in how students explain their thinking, and explore how those ideas connect and are visualized through other students' representations.

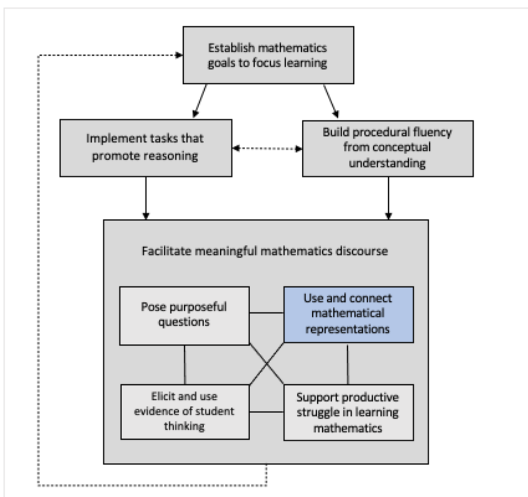
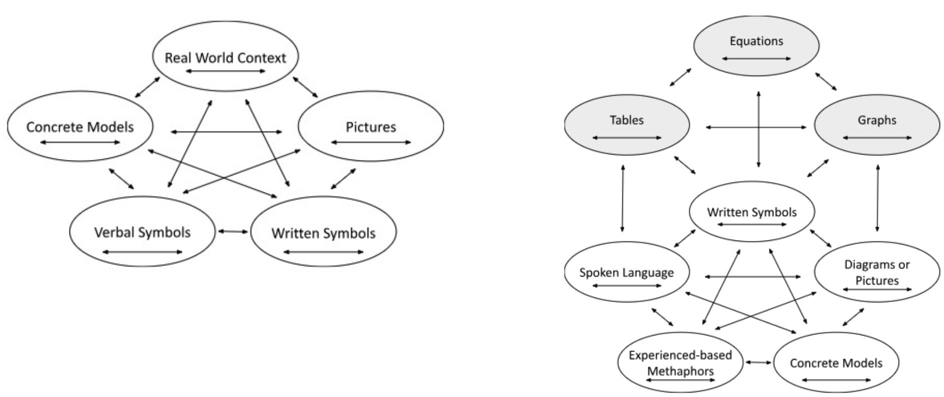


Figure 1
Eight Mathematical Teaching Practices
Principles to Actions, NCTM 2024, 2014

Capturing mathematical ideas through the use of various models results in being able to consider and analyze the relationships and the procedural actions used to find a solution(s). Models capture the strategies used. Seeing the mathematics modeled in a variety of representational forms allows learners to consider the connections and relationships among them. It allows thinking to be extended and considered by others as the ideas are communicated publicly. The ability to flexibly move across representations is critical as each model displays the mathematical concepts differently allowing the full range of learners within the space to comprehend and connect important ideas.

Many representations are social conventions that have emerged over time to reflect the consensus over time of the mathematical community. Equations, graphs, and tables follow certain conventions. These conventions can be directly instructed. However, it is well documented that learners create their own mathematical ways of conveying their ideas that are quite inventive and powerful in conveying their thinking. As the classroom teacher, when and how to introduce certain mathematical representational conventions, along with ***attending to*** and ***interpreting*** student modeling, requires an intentionality that influences ***deciding how to respond*** to students' thinking.

A model captures visually the strategy used to solve a task. Models can range from the informal to the very formal. Models can be drawn, pictorial, graphical, and numerical. Drawing students' attention to connections between and among different models aids students in comprehending the mathematical relationships captured differently within each. A student who has written an equation can be asked how their equation is captured in the table created by another student. Pressing students to provide the details of their thinking as well as considering the details of their peers' work has been shown to have a positive effect on student achievement (Bishop, 2021; Webb, et al., 2014; Kazemi & Stipek, 2001).



[a] [b]

Figure 2a – *Lesh Translation Model (1979)*

Figure 2b – *Kaput-Lesh merged translation model (Kaput, 1987; Lesh & Doerr, 2003)*

The Lesh translation model, first articulated in 1979 , and expanded later in 1987 by Kaput, serves as a means to understand how different neurological pathways can be activated as one moves within and between different representations. These two models are an outgrowth of the more linear C-R-A model, Concrete-Representational-Abstract sequence proposed by Bruner and Kenney (1965). The Lesh Translation Model allows for the flexibility of entry points depending upon what the student already knows rather than in a predetermined linear sequence. One could use the C-R-A sequence, but other options are possible. Knowing what the student already knows is key to understanding how one might proceed.

Think about how a task may be presented to students through a real-world context or as a non-context-based true/false equation, for example. As students work towards a solution they may draw a picture, use manipulatives, verbalize their rationale for their chosen process. For the individual students, each time they move from one representation to another more neurological pathways are activated. Publicly during sharing, the teacher, by cognitively guiding the conversation through the strategic use of prompts and questions, stimulates students to both reflect on their own personal decision making as well as drawing attention to the representations used by their peers. Pressing for details and comparing and contrasting how the work is captured across the various representations deepens the comprehension, connects mathematical ideas together, draws attention to efficiencies and standardization of notation, and moves the mathematics forward vertically.

At times the teacher serves the role of translator or secretary. The student talks while the teacher captures the work publicly for all to see. As students express their thinking, the teacher is in the position of selecting which representation to use to capture that thinking. If the task involves one of the four operations, should the recording be in vertical or horizontal formatting or both? Connecting to manipulatives or a picture? Would the mathematical ideas be better presented on a number line? In a table? On a graph? A matrix? Or in a combination of representations appropriate to the task. Some students better understand the mathematical relationships if they work within a table first before moving on to an equation or graph. Others may find graphing first aids in understanding the mathematical relationships. Key to allowing for multiple representations is that, as a skillfully facilitated conversation unfolds within the learning space, students come to see the connections among the table, the graph, and the equation.

Consider some of the following representations for a discussion around various fraction ideas.

Put these numbers in order from largest to smallest.

$$\frac{2}{6} \quad \frac{1}{5} \quad \frac{3}{8}$$

Some students might verbalize the number sense that reasons the seriation choices that they made. This reasoning can be potentially captured as a series of equality and inequality expressions. Some students might free-sketch pictures; others may work on a number line. Some students might utilize digital tools such as the fraction bars available on Polypad or at the Math Learning Center. In having all representations publicly shared, the discussion can focus on the strategies and analysis used in common among all. Efficiencies can be explored. New examples can be given to have students use ideas from peers to save time while working with the new set of numbers. Targeted strategic scaffolds from the teacher can elevate the strategies explored by those specific students who may need emotional support to try a new, more abstract level strategy. In working across the representations, having students see their thinking in the other student's representations deepens all students' comprehension and the possibility for taking on more efficient representations and levels of reasoning in future contexts.

Examples of reasoning and representations

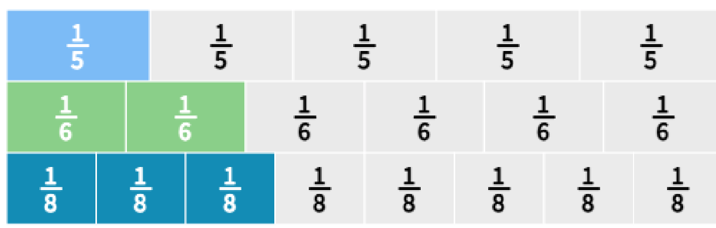
Student A: Using abstract number strategies and logic

All three are less than $\frac{1}{2}$ because $\frac{1}{2} = \frac{3}{6} = \frac{2.5}{5} = \frac{4}{8}$

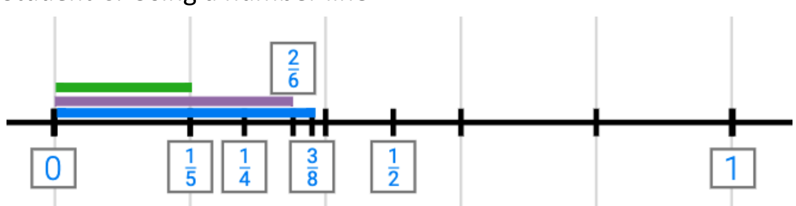
$\frac{2}{6}$ is $\frac{1}{6}$ less than $\frac{1}{2}$ $\quad$ $\frac{3}{8}$ is $\frac{1}{8}$ less than $\frac{1}{2}$ $\quad$ $\frac{1}{8} < \frac{1}{6}$ so $\frac{3}{8} > \frac{2}{6}$ $\quad$ [Residue strategy]

$\frac{2}{6} = \frac{1}{3}$ $\quad$ $\frac{1}{3} > \frac{1}{5}$ $\quad$ So $\frac{1}{5} < \frac{2}{6} < \frac{3}{8}$

Student B: Using fraction bars



Student C: Using a number line



[Teacher Prompt]: Student A, where in the number line or fraction bars do you see your reference of $\frac{1}{2}$?

[Teacher Prompt]: Student B, where is the $\frac{1}{2}$ reference that Student A used in your fraction bars?

[Teacher Prompt]: Class, Student A made the statement that $\frac{1}{8} < \frac{1}{6}$ so $\frac{3}{8} > \frac{2}{6}$. Talk with a partner. Justify that statement using what we see in either Student B or C's representation. Be prepared to explain you and your partner's reasoning when we come back together.

Using the work of different students allows the ideas of one student to help all in the classroom use evidence to explore the reasoning of those sharing and strengthen their own number sense and improve one's ability to progress towards more abstract thinking.

Staying with fraction examples, consider the shaded area in the grid.

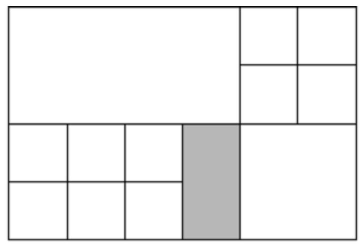


Figure 3
From the work of S. Lamon, 2010

[Teacher prompt]: *Without counting by ones, how much of the whole outside rectangle is shaded? And how did you reason through to your solution?*

[Teacher Prompt]:

- *Talk to a partner, how do you see the shaded area as* $\frac{1}{4}$ *of a* $\frac{1}{3}$ *piece? Be prepared to VERBALLY describe the position on the grid to justify with evidence your reasoning.*
- *… a* $\frac{1}{6}$ *of a* $\frac{1}{2}$ *piece?*
- *…* $\frac{1}{2}$ *of a* $\frac{1}{6}$ *piece?*

[Teacher Follow up Prompt]:

- *True or false? If true, why true? If false, why false?*
  - $\frac{1}{4} x \frac{1}{3} = \frac{1}{6} x \frac{1}{2} = \frac{1}{12} x\ 1$
- *Using the grid representation, how is the relationship captured in the expression* $\frac{1}{6} x \frac{1}{2}$ *different from* $\frac{1}{2} x \frac{1}{6}$*?*

Using and connecting mathematical grids is integrally enhanced with the intertwining of the types of prompts and questions posed to elicit students' thinking. Being familiar with various representations by which the mathematical concepts and relationships can be displayed, both spatial and numerical, is critical for all teachers to be comfortable with. Varying the representations allows students to contemplate mathematics through different perspectives. What connections one student sees in one representational format may be different from their counterparts. By varying the representations, whether informal or formal, students deepen their understanding as they progress towards working with standard notation formats. Reasoning needs to be nurtured. The language to express one's reasoning needs to be systematically supported. Forms of mathematical representation and notation are social conventions that mathematicians have built a consensus around. Those can be explicitly introduced. Reasoning needs to be scaffolded and nurtured.

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